A geometric approach to D-branes in group manifolds [1]

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Abstract: This is a brief review[2] of some recent results on the geometric approach to symmetric D-branes in group manifolds, both twisted and untwisted. We describe the geometry of the gluing conditions and the quantisation condition in the boundary WZW model, and we illustrate this by determining the consistent twisted and untwisted D-branes in the Lie group SU_3.

1 Introduction

D-branes in group manifolds have attracted a great deal of attention in recent years, as they provide an ideal laboratory for the study of D-branes in general string backgrounds. Using a variety of approaches, ranging from the algebraic techniques of BCFT to the lagrangian description based on the boundary WZW model, it has been possible to analyse in a detailed and systematic fashion what are the consistent D-brane configurations in a given group manifold and how they can be classified.

In this talk, based on [1, 2, 3, 4, 5], we present a geometric approach to the study of D-branes in group manifolds (that is, in WZW models). In Section 2 we describe how the classical geometry of these D-branes can be determined directly from the gluing conditions. In Section 3 we discuss the boundary WZW model and how it can be thought of as providing a lagrangian description of D-branes in group manifolds. In particular, we exhibit the two-form field defined on the D-brane and the quantisation conditions obtained by requiring that the path integral be well defined. Finally, in Section 4, we describe the classical and quantum moduli spaces of consistent D-brane configurations in the Lie group SU_3. In particular, we show that (twisted) D-brane configurations are in one-to-one correspondence with the integrable highest weight (IHW) representations of the (twisted) affine Lie algebra $\widehat{\mathfrak{su}}(3)^{(2)}_k$.

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2 Symmetric D-branes in group manifolds

The simplest and best understood class of D-brane configurations is obtained [6, 7, 1] as solutions of the familiar gluing conditions on the chiral currents of the WZW model

$$J(z) = R\bar{J}(\bar{z}) \qquad \text{at the boundary,} \tag{1}$$

where R is a metric preserving Lie algebra automorphism. These gluing conditions describe *symmetric* D-branes, that is, configurations which preserve the maximal amount of symmetry of the bulk theory; that is, conformal invariance plus (some of) the current algebra.

Let us assume, for simplicity, that G is a compact, connected, simply-connected Lie group. A D-brane in G wraps a submanifold Q of G on which there is defined a two-form field ω. The D-submanifold Q can be determined directly from the gluing conditions (1), by using the following geometric interpretation [1]. The boundary conditions satisfied by an open string whose end lies on Q are given by

$$\partial g = \tilde{R}(g)\bar{\partial} g \ ,$$

where $\tilde{R}(g) = \rho_g R \lambda_{g^{-1}}$ is the point dependent matrix of boundary conditions. The tangent space to the D-brane T_gQ and its perpendicular complement $T_gQ^{\perp}$ are spanned by the eigenvectors of $\tilde{R}(g)$ corresponding to the Neumann and Dirichlet conditions, respectively. In particular, one can show that the tagent space to the D-brane

$$T_gQ = \mathrm{Im}(\mathbb{1} + \tilde{R}(g)) \ ,$$

is nothing but the tangent space $T_g\mathcal{C}_r(g)$ to a twisted conjugacy class [1, 7]

$$\mathcal{C}_r(h) = \left\{ r(g)hg^{-1} \mid g \in G \right\} \ ,$$

where $r : G \to G$ is the Lie group automorphism induced by R. This shows that D-branes described by (1) wrap twisted conjugacy classes in the group manifold G. In the special case $R = \mathbb{1}$, one obtains [6] the standard conjugacy classes of G.

The twisted conjugacy class $\mathcal{C}_r(h)$ of an element h of G is defined as the orbit of h in G under the twisted adjoint action $\mathrm{Ad}^r_g : g \mapsto r(g)hg^{-1}$, for any g in G. Since the stabiliser of h is given, in this case, by its twisted centraliser $\mathcal{Z}_r(g) = \{g \in G \mid r(g)h = hg\}$, the twisted conjugacy class $\mathcal{C}_r(h)$ can be described as the homogeneous space

$$\mathcal{C}_r(h) \cong G/\mathcal{Z}_r(g) \ . \tag{2}$$

3 The boundary WZW model

The boundary WZW [8, 9, 3] model can be thought of as a lagrangian description for D-branes in group manifolds. In this framework, a D-brane is described by a submanifold $\iota : Q \to G$ together with a two-form ω on Q such that $\iota^* H = d\omega$, where $H = 1/6 \ \langle \theta, [\theta, \theta] \rangle$ denotes the three-form field on the target group manifold, θ is the left-invariant Maurer-Cartan one-form on G, and $\langle -, - \rangle$ is an invariant metric on the Lie algebra $\mathfrak{g}$ of G. The classical dynamics of an open string whose end lies on this D-brane is governed by the action

$$I = \int_{\Sigma} \langle g^{-1}\partial g, g^{-1}\bar{\partial} g \rangle + \int_M H - \int_D \omega \ , \tag{3}$$

where M is a 3-dimensional submanifold of G with boundary $\partial M = g(\Sigma) + D$, and D is a 2-dimensional submanifold of Q. There exists a homological obstruction [3] to the existence of M which is measured by the relative homology class of $g(\Sigma)$ in $H_2(\mathbf{G}, Q)$.

The action of the boundary WZW model (3) is constructed as a natural generalisation of the standard WZW action; in particular, it is invariant under the infinite-dimensional symmetry group generated by the transformations

$$g(z,\bar{z}) \mapsto \Omega(z) g(z,\bar{z}) \bar{\Omega}(\bar{z})^{-1} \ . \tag{4}$$

The parameters $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ of these transformations satisfy $\bar{\partial}\Omega = \partial\bar{\Omega} = 0$ and are such that (4) preserves the worldsheet boundary $g(\Sigma) \subset \mathcal{C}_r$; this latter property is encoded in the condition $\Omega(z) = r \cdot \bar{\Omega}(\bar{z})$, at the boundary $\partial\Sigma$. In terms of the conserved chiral currents $J(z)$ and $\bar{J}(\bar{z})$, this gives rise to the gluing conditions (1).

The two-form field ω is uniquely determined by the symmetry requirement (4), being given [2] (see also [6, 9] for the case $R = \mathbb{1}$) by

$$\omega = -\tfrac{1}{2} \, \langle g^{-1}dg \ , \frac{\mathbb{1} + \mathrm{Ad}_{g^{-1}} R}{\mathbb{1} - \mathrm{Ad}_{g^{-1}} R} \ g^{-1}dg \rangle \ . \tag{5}$$

One can easily check that the right hand side is well defined on the twisted conjugacy class $\mathcal{C}_r$, and that (5) defines a field which satisfies $d\omega = \iota^* H$.

In the case of the standard WZW model, we recall that the cancellation of the global worldsheet anomaly imposes that the period of the three-form field H over any 3-cycle in $H_3(\mathbf{G})$ be quantised, that is $[H]/2\pi \in H^3(\mathbf{G}, \mathbb{Z})$. Similarly, in the case of the boundary WZW model, the condition that the path integral be single valued (that is, that it not depend on the choice of M) imposes that the global worldsheet anomaly vanish, which translates into

$$\int_N H - \int_{\partial N} \omega \ \in 2\pi\mathbb{Z} \ , \tag{6}$$

for any relative 3-cycle $(N, \partial N)$ in $H_3(G, \mathcal{C})$. In other words, $[(H, \omega)]/2\pi$ must define a class in $H^3(\mathbf{G}, \mathcal{C}_r; \mathbb{Z})$.

4 Example: Symmetric D-branes in SU_3

Let us now apply the formalism described in the previous sections in order to determine the consistent symmetric D-branes in SU_3, for which we have two distinct classes of solutions: untwisted branes, characterised by $r = \mathbb{1}$, and twisted branes, defined by $r = \tau$, where τ is the Dynkin diagram automorphism of SU_3.

4.1 Untwisted branes

Conjugacy classes in a group G are parametrised by the maximal torus T of G, modulo the Weyl group. Standard group theory tells us that the classical moduli space of D-branes in SU_3, denoted here by $\mathcal{M}_{cl}(\mathrm{SU}_3, \mathbb{1})$, which is the same as the space of conjugacy classes of SU_3, can be identified with the fundamental domain of the extended Weyl group in the Cartan subalgebra $\mathfrak{t}$, which is given by the (solid) equilateral triangle

$$\mathcal{M}_{cl}(\mathrm{SU}_3, \mathbb{1}) = \{ X \in \mathfrak{t} \mid 0 \leq \alpha_i(X) \leq 1 \ , \ i = 1,2,3 \} \ , \tag{7}$$

where α_1, α_2 and $\alpha_3 = \alpha_1 + \alpha_2$ are the positive roots of SU_3. The interior points in $\mathcal{M}_{cl}$ are regular, and give rise to 6-dimensional conjugacy classes of the form $\mathrm{SU}_3/\mathrm{U}_1^2$. If we now consider an element X in $\mathfrak{t}$ which belongs to one of the edges of $\mathcal{M}_{cl}$, this describes an element $h = \exp(X)$ in the maximal torus of SU_3, whose centraliser includes a $\mathrm{SU}_2^{\alpha_i}$ subgroup, for some i. Thus the boundary points belonging to the three edges are singular, giving rise to 4-dimensional conjugacy classes of the form $\mathrm{SU}_3/\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_1)$. Finally, the three vertices corresponding to the three central elements of SU_3 describe point-like D-branes of the form $\mathrm{SU}_3/\mathrm{SU}_3$. We thus obtain the space of classical symmetric D-branes represented in Figure 1.

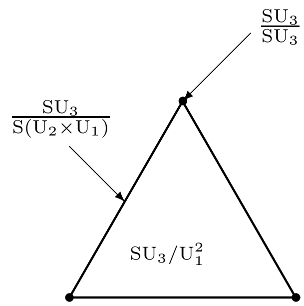

Figure 1: Moduli space of conjugacy classes of SU_3

By working out the quantisation conditions (6) one obtains that the quantum moduli space $\mathcal{M}_q(\mathrm{SU}_3, \mathbb{1})$ of D-branes in SU_3 at level k is given by

$$\mathcal{M}_q(\mathrm{SU}_3, \mathbb{1}) = \{X \in \mathfrak{h} \mid k\alpha_i(X) \in \mathbb{Z},\ 0 \leq k\alpha_i(X) \leq k,\ i = 1,2,3\}\ , \tag{8}$$

which proves [9] that the set of consistent symmetric D-brane configurations at level k is in one-to-one correspondence with the set of IHW representations of the corresponding affine Lie algebra $\widehat{\mathfrak{su}}(3)_k^{(1)}$.

The space of untwisted D-brane configurations in SU_3 for the first few values of the level k is represented in Figure 2. At a given level k we have 3 point-like, $3(k-1)$ 4-dimensional and $\frac{1}{2}(k-1)(k-2)$ 6-dimensional symmetric D-branes. We also see that the 4-dimensional conjugacy classes are characterised by quantum numbers (λ_1, λ_2) with either one of the λ's being equal to zero or $\lambda_1 + \lambda_2 = k$; the point-like conjugacy classes are described by $(0,0)$, $(0,k)$, $(k,0)$. In particular, the lower-dimensional conjugacy classes dominate the spectrum of D-branes until $k = 9$.

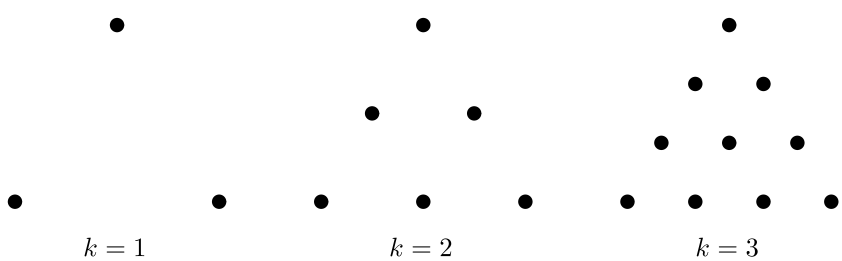

Figure 2: Quantum moduli space for SU_3 for lowest values of the level k.

4.2 Twisted branes

Let us now turn to the case of twisted branes. Twisted conjugacy classes are parametrised by the maximal torus T^τ of the fixed point subgroup $\mathrm{SU}_3^\tau \cong \mathrm{SO}_3$, modulo the twisted Weyl group W_τ of SU_3. In order to understand W_τ and determine the space of twisted conjugacy classes, one needs to make an incursion [12, 13] into the theory of non-connected Lie groups. One obtains [4] in this fashion a nice description of the classical moduli space of twisted branes in SU_3 in terms of the classical moduli space of untwisted branes in SU_3^τ. More precisely, we have

$$\mathcal{M}_{cl}(\mathrm{SU}_3, \tau) = \left\{ X \in \mathfrak{t}^\tau \mid 0 \leq \bar{\alpha}(X) \leq \tfrac{1}{4} \right\} \ , \tag{9}$$

where $\bar{\alpha} = \frac{1}{2}(\alpha_1 + \alpha_2)$ is the root of SU_3^τ.

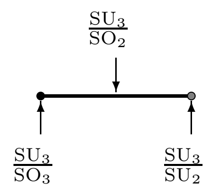

Figure 3: Moduli space of twisted conjugacy classes of SU_3

The resulting space of classical twisted D-branes in SU_3 is described in Figure 3. The point $\bar{\alpha}(X) = 0$ is singular in SO_3 and τ-singular (see [4]) in SU_3, giving rise to a 5-dimensional twisted D-brane of the form $\mathrm{SU}_3/\mathrm{SO}_3$. The other endpoint $4\bar{\alpha}(X) = 1$ is regular in SO_3, but τ-singular in SU_3. The corresponding twisted class is also 5-dimensional, but has the form $\mathrm{SU}_3/\mathrm{SU}_2$. Finally, the interior points are τ-regular and give rise to 7-dimensional twisted conjugacy classes of the form $\mathrm{SU}_3/\mathrm{SO}_2$. Notice that in this case we have that the dimension of the twisted conjugacy classes is always odd, due to the fact that the difference between the ranks of SU_3 and SO_3 is odd.

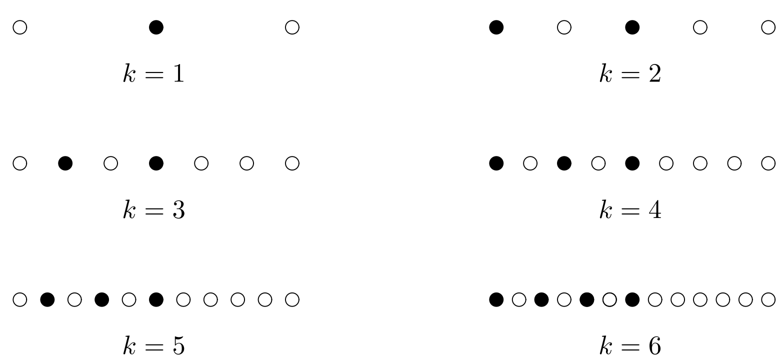

Figure 4: Quantum moduli space for τ-twisted D-branes in SU_3 at level k (black circles) compared with that of $\mathrm{SU}_3^\tau \cong \mathrm{SO}_3$ (white circles) at level $4k$.

If we now impose the condition (6) for the cancellation of the global worldsheet anomaly, we obtain that the quantum moduli space of twisted D-branes in SU_3 is given by

$$\mathcal{M}_q(\mathrm{SU}_3,\tau) = \begin{cases} \{X \in \mathfrak{t}^\tau \mid 4k\bar{\alpha}(X) = 1,3,...,k\} \ , & \text{for} \quad k \text{ odd} \ , \\ \{X \in \mathfrak{t}^\tau \mid 4k\bar{\alpha}(X) = 0,2,...,k\} \ , & \text{for} \quad k \text{ even} \ . \end{cases} \tag{10}$$

The states corresponding to the first few values of the level are represented in Figure 4. At a given odd level k we have $\frac{1}{2}(k-1)$ 7-dimensional and one 5-dimensional branes, whereas for k even we have $(\frac{1}{2}k-1)$ 7-dimensional and two 5-dimensional branes.

A careful comparison of the quantisation conditions (6) for the twisted branes with the spectrum [11] of IHW representations of the twisted affine Lie algebra $\widehat{\mathfrak{su}}(3)^{(2)}_k$ reveals [4] that the admissible twisted D-brane configurations in SU_3 are in one-to-one correspondence with the IHW representations of the corresponding twisted affine Lie algebra $\widehat{\mathfrak{su}}(3)^{(2)}_k$.

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